



Introduction and Growth Rates

Statistics I

Course Introduction

Core Information

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Main source for course material: Moodle@ISCAL

Official communication channel:  Institutional Email

Bibliography

Topic 1:

 Gancho Custódio, S. et al. (2022) Números Índices, Edições Sílabo

Topic 2:

 Murteira, B.; Silva Ribeiro, C.; Andrade e Silva, J. & Pimenta, C.,
Introdução à Estatística, Escolar Editora, McGraw-Hill, 2010

Topic 3:

 Ferreira, T., Custódio, S.G. (2023) Modelos Probabilísticos, Edições
Sílabo

Bibliography (EN)

Topic 1:

 The making of index numbers. 1st Ed 1922. Irving Fisher.

 A Practical Introduction to Index Numbers. 1st Ed 2015. Jeff Ralph, Rob O'Neill, Joe Winton.

Topics 2 and 3:

 Statistics for Business and Economics Global Edition. 10th Ed 2022. Paul Newbold, William Carlson, and Betty Thorne.

This last book is a bit expensive, but it can be rented for less money [here](#).

Continuous Assessment

Assessment Element	Weight (%)	Duration	Syllabus	Date
Midterm 1	60%	80 minutes	Topics 1 and 2	Nov 5 th
Midterm 2	40%	80 minutes	Topic 3	Dec 10 th

Each test has a minimum grade of 7.00 (out of 20), not 6.9, not 6.95.

Comprehensive Exam

Students may opt, on the day of the Exam, to take an Exam worth 100% of the grade. This Exam will cover Topics 1, 2, and 3.

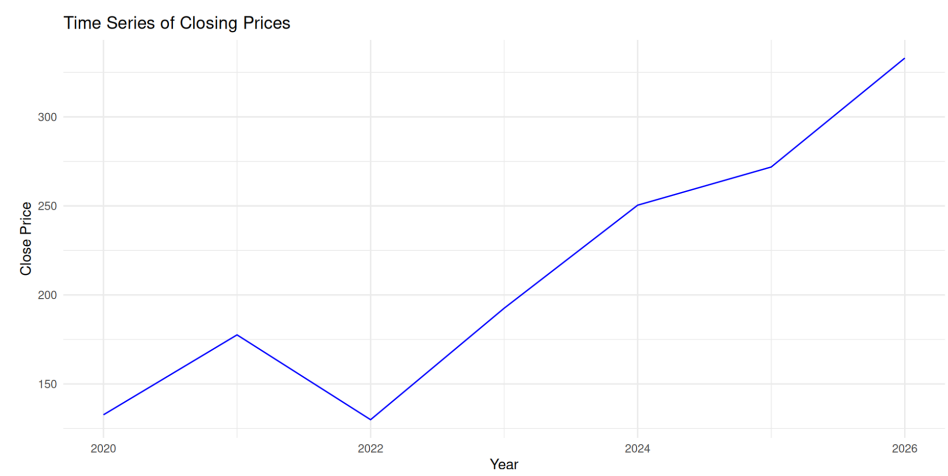
If you score less than 7.00 in the Midterm, or if you cannot attend the midterm for whatever reason, you **will** have to do the Comprehensive Exam.

Disclaimer

These slides are a free translation and adaptation from the slide deck for Estatística I by Prof. Sandra Custódio and Prof. Teresa Ferreira from the Lisbon Accounting and Business School, Polytechnic University of Lisbon.

Growth Rates

Example: Apple Stock



Source: Yahoo Finance

Date	Close
2020	132.69
2021	177.57
2022	129.93
2023	192.53
2024	250.42
2025	271.86
2026	333.02

Example: Apple Stock

Let the close price for Apple be represented by the variable y_t , so y_{2020} is the close price we got for 2020 (*i.e.* 132.69).

How much did y grow between 2020 and 2021?

$$\begin{aligned}\Delta y_{2021} &= y_{2021} - y_{2020} = 177.57 - 132.69 \\ &= 44.88\end{aligned}$$

Or we could say

$$\begin{aligned}y_{2021} &= y_{2020} + \Delta y_{2021} \\ 177.57 &= 132.69 + 44.88\end{aligned}$$

Example: Apple Stock

$$y_{2021} = y_{2020} + \Delta y_{2021}$$

$$y_{2021} = y_{2020} \left(1 + \frac{\Delta y_{2021}}{y_{2020}} \right)$$

$$y_{2021} = y_{2020} (1 + \delta_{2021}^y)$$

This, δ_{2021}^y , is the **growth rate** of y at year 2021.

Growth Rate

Growth Rate

Let y_t be a variable that might take different values over time. The **growth rate** at t is given by δ_t^y , and takes a value such that:

$$y_t = y_{t-1} (1 + \delta_t)$$

And then

$$\delta_t = \frac{y_t - y_{t-1}}{y_{t-1}}$$

Note: I dropped y from the notation in the formula because it is obvious that we are talking about the variable y .

Growth Rate

You can write the growth rate as a decimal or as a percentage. If you use decimal notation, consider at least 4 places, if you use percentage you use at least 2. Example: 0.0123 or 1.23%.

Interpretation: What is the percentage change for the variable y between $t - 1$ and t .

Example

$$177.57 = 132.69 + (177.57 - 132.69)$$

$$177.57 = 132.69 \left(1 + \frac{44.88}{132.69} \right)$$

$$177.57 = 132.69 (1 + 0.3382)$$

In this case $\delta_{2021}^y = 0.3382 = 33.82\%$.

Interpretation: Between 2020 and 2021, the price for Apple *increased* 33.82%.

Growth Rates

More generally

$$\delta_{t+k|t} = \frac{y_{t+k} - y_t}{y_t}$$

Or

$$y_{t+k} = y_t (1 + \delta_{t+k|t})$$

(homework: Show step by step how you go from one to the other.)

Note: In this case $\delta_{t+k|t}$ is what percentage the variable changed from t to $t + k$, and k is the number of periods.

Example

We had $y_{2020} = 132.69$, and $y_{2024} = 250.42$. We will try to find $\delta_{2024|2020}^y$

- What is the value of k in this case?

$$k = 2024 - 2020 = 4$$

- What is the growth rate between 2020 and 2024?

$$\delta_{2024|2020}^y = \frac{250.42 - 132.69}{132.69} = 0.8873 = 88.73\%$$

For Apple, the stock price grew 88.73% between 2020 and 2024, or the stock price for Apple in 2024 is 88.73% larger than in 2020.

Growth Rates

Warning

You should be careful with your interpretation and use of this *cumulative* growth rate, because it **does not** mean that y grew $\delta_{t+k|t}^y$ **per period** between t and $t + k$.

Growth Rates

Average Growth Rate

Let y_t be a variable that might take different values over time. The **average growth rate** between t and $t + k$ is given by $\bar{\delta}_{t+k|t}^y$, and takes a value such that:

$$y_{t+k} = y_t \left(1 + \bar{\delta}_{t+k|t}\right)^k$$

We will use something called the *geometric mean*.

I dropped y from the notation in the formula because it is obvious that we are talking about the variable y .

Growth Rates

Let's start with $y_t \dots$ and the traditional growth rates δ_t :

$$y_{t+1} = y_t(1 + \delta_{t+1})$$

$$y_{t+2} = y_{t+1}(1 + \delta_{t+2})$$

$$y_{t+3} = y_{t+2}(1 + \delta_{t+3})$$

Can we write y_{t+2} as a function of y_t ?

$$y_{t+2} = y_{t+1}(1 + \delta_{t+2}) = y_t(1 + \delta_{t+1})(1 + \delta_{t+2})$$

And y_{t+3} ?

$$y_{t+3} = y_t(1 + \delta_{t+1})(1 + \delta_{t+2})(1 + \delta_{t+3})$$

Growth Rates

Generalizing

$$y_{t+k} = y_t (1 + \delta_{t+1}) \dots (1 + \delta_{t+k-1}) (1 + \delta_{t+k})$$

When computing the average growth rate, we are trying to find $\bar{\delta}$ such that if the growth rate was the same every period, it would have taken y_t to the value of y_{t+k} all the same:

$$y_{t+k} = y_t (1 + \bar{\delta})(1 + \bar{\delta}) \dots (1 + \bar{\delta})$$

How many times is $(1 + \bar{\delta})$ multiplied in the expression?

Growth Rates

$$y_{t+k} = y_t \left(1 + \bar{\delta}\right)^k$$

$$\frac{y_{t+k}}{y_t} = \left(1 + \bar{\delta}\right)^k$$

$$\left(\frac{y_{t+k}}{y_t}\right)^{1/k} = \left(1 + \bar{\delta}\right)$$

$$\left(\frac{y_{t+k}}{y_t}\right)^{1/k} - 1 = \bar{\delta}$$

$$\bar{\delta}_{t+k|t} = \left(\frac{y_{t+k}}{y_t}\right)^{1/k} - 1$$

Example

We had

$$\delta_{2024|2020}^y = \frac{250.42 - 132.69}{132.69} = 0.8873 = 88.73\%$$

But what now is the **average growth rate** between 2020 and 2024?

Remember $k = 4$

$$\bar{\delta} = \left(\frac{250.42}{132.69} \right)^{1/4} - 1 \approx 0.1721 = 17.21\%$$

Interpretation: On average, between 2020 and 2023 y grew 17.21% every year.

Example

Note that

$$(1 + 17.21\%)^4 = (1 + 0.1721)^4 \approx 1.8873 \approx 1 + 88.73\%$$

? Growth Rates · Question 1

A variable grows 10% in one year and then falls 10% in the next. Over the two years together it:

- A. fell 1%
- B. rose 1%
- C. is unchanged
- D. fell 10%



A. Growth rates compound, they do not add:

$(1 + 0.1)(1 - 0.1) = 0.99$, so the variable ends 1% **below** where it started.

? Growth Rates · Question 2

A variable goes from 100 to 150 over 5 years. Its **average** growth rate is closest to:

A. 10%

B. 8.45%

C. 50%

D. 5%

☒ **B.** $\bar{\delta} = \left(\frac{150}{100}\right)^{1/5} - 1 = 1.5^{0.2} - 1 \approx 0.0845$, that is 8.45% per year.



Growth Rates · Question 3

A firm sold 240 units in 2020 and 300 units in 2023.

Compute:

- (i) the cumulative growth rate over the three years;
- (ii) the average annual growth rate.



Growth Rates · Solution

(i) $\delta_{2023|2020} = \frac{300-240}{240} = 0.25$, a cumulative 25%.

(ii) Here $k = 3$, so $\bar{\delta} = \left(\frac{300}{240}\right)^{1/3} - 1 = 1.25^{1/3} - 1 \approx 0.0772$, about **7.72% per year**.

Note 7.72% per year for 3 years is *not* 25/3, precisely because growth compounds.

Index Numbers: Introduction and Simple Index Numbers

Motivation



Which stock would you have purchased in 2016-10-03?

Motivation

What's the important question here?

Which one grew more! For this, levels are not as relevant as their evolution over time.

For example, say we have prices for two stocks A and B. Say we want to hold these stocks for only one period, which one would choose to invest your hard earned 💰?

How many stocks will you be able to buy of each stock, if their prices are a_t and b_t , and you have US\$1000?

$n_a = \frac{1000}{a_t}$ of stock A and $n_b = \frac{1000}{b_t}$ of stock B. Note that n_a and n_b are the amount of stock you can buy of each.

Motivation

Now is a new day ! Prices are now a_{t+1} and b_{t+1} , how much is your portfolio worth today?

1. If you bought stock A: $a_{t+1}n_a = a_{t+1} \frac{1000}{a_t}$
2. If you bought stock B: $b_{t+1}n_b = b_{t+1} \frac{1000}{b_t}$

Note that the 1000 is fixed, so it is just what we invested, it is not going to change anything in our decision on buying A vs B, what is really important is $\frac{a_{t+1}}{a_t}$ and $\frac{b_{t+1}}{b_t}$, that is the growth rate of prices for each stock, as

$$\frac{a_{t+1}}{a_t} = 1 + \delta_{t+1}^a \quad \text{and} \quad \frac{b_{t+1}}{b_t} = 1 + \delta_{t+1}^b$$

Motivation

You did not sell and...

Now is (another) new day ! Prices are now a_{t+2} and b_{t+2} , how much is your portfolio worth today?

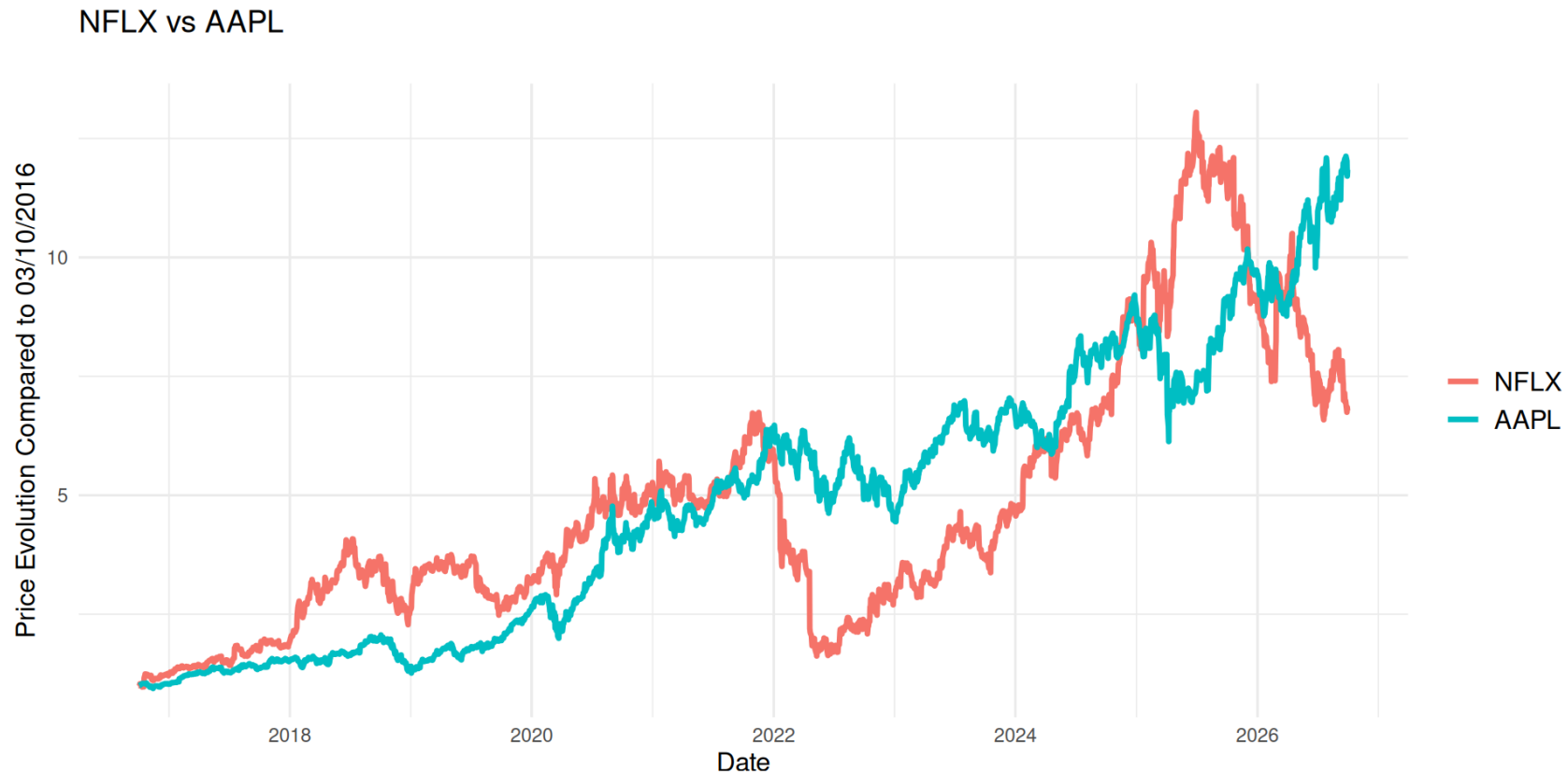
1. If you bought stock A: $a_{t+2}n_a = a_{t+2} \frac{1000}{a_t}$
2. If you bought stock B: $b_{t+2}n_b = b_{t+2} \frac{1000}{b_t}$

blah, blah...

$$\frac{a_{t+2}}{a_t} = 1 + \delta_{t+2}^a \quad \text{and} \quad \frac{b_{t+2}}{b_t} = 1 + \delta_{t+2}^b$$

If we wait two days, we only care how is the price today compared to when we made the purchase. We could make instead a plot of $1 + \delta_{t+k}$ to see which grew more!

Motivation



Now we have a much clear picture. We can compare their evolution starting in 2016-10-03.

What can you read from the plot?

Motivation

Note that for 2016-10-03, we would have, for A and B:

$$1 + \frac{a_t - a_t}{a_t} = 1 \quad \text{and} \quad 1 + \frac{b_t - b_t}{b_t} = 1$$

We just built an index number!

By convention though, instead of using 1 for the reference date, we will use 100 (as in 100%) for the reference day (we will put a name to this in a few moments).

When our new series was 2, we understood that $1 + \delta_{t+k} = 2$ or $\delta_{t+k} = 1$, *i.e.* the price grew 100%, or *doubled*! Now, with the new notation, we would have that the new series would take the value of 200, *i.e.* the price is now 200% the value it had at our initial date.

Index Numbers

Index Number

An **Index Number** is a scaled variable that takes as a reference point the value an underlying variable took at a fixed point in time. Say, for an underlying variable x_t we build an index I_t .

$$I_t = \frac{x_t}{x_0} \times 100$$

Where x_t is the value the underlying variable takes at time t , and x_0 represents the value this variable took at our reference point. This period is called **base**.

Note: In this case we used time as the dimension for t but this needs not be the case, we will see examples later.

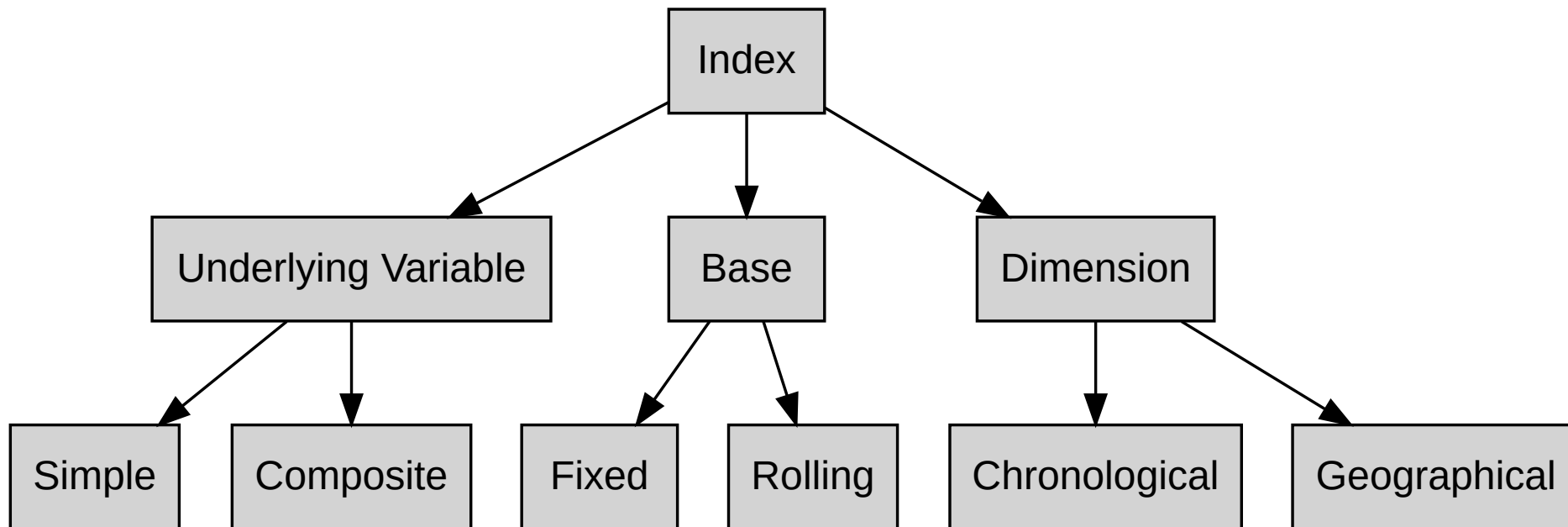
In our previous example, the base day would have been 2016-10-03. Note that in the base period

$$I_0 = \frac{x_0}{x_0} \times 100 = 100$$

Index Numbers

Index Numbers are defined by their underlying variable. This can change over time, or it can change according to another dimension, like geography, or both! We will see examples.

Also, an Index Number can have a fixed base or a moving base.



Index Numbers



Index Numbers

- Simple and Composite are mutually exclusive
- Fixed and Moving are mutually exclusive
- Rolling always needs a chronological dimension

Examples

Type	Example
Simple	Single stock price
Composite	SP500 (market index), CPI
Fixed Base	GDP, CPI, market index
Rolling Base	Some measures of GDP, Commodities Futures
Chronological	Stock price over time, CPI, GDP
Geographical	BigMac Index

Index Numbers

Importantly, an index number is *no bueno* to reflect the level of a variable, but its evolution or relative value against a base.

Rolling Base

Chain or Link Index

A **Chain or Link Index Number** is an index number that updates its base in every period.

$$I_t = \frac{x_t}{x_{t-1}} \times 100$$

Interpretation: It tells you how much the variable increased compared to the **previous** period.

Index Numbers

Index numbers allow us to easily compute the growth rate between any two periods, just like we would do with the underlying variable.

1. Simple Index Number

$$1 + \delta_{t+k,t} = \frac{x_{t+k}}{x_t} = \frac{\frac{x_{t+k}}{x_0}}{\frac{x_t}{x_0}} = \frac{I_{t+k}}{I_t}$$

Index Numbers

2. Chain Index Number

$$1 + \delta_{t+k,t} = \frac{x_{t+k}}{x_t} = \frac{x_{t+k}}{x_{t+k-1}} \frac{x_{t+k-1}}{x_{t+k-2}} \dots \frac{x_{t+1}}{x_t}$$

$$1 + \delta_{t+k,t} = I_{t+k} \times I_{t+k-1} \times \dots \times I_{t+1}$$

Index Numbers



Do not use 100 index when multiplying chain or link indices, use the decimal notation.

Note that if the underlying variable does not change at all, you should expect the same value, $x_t = x_{t+1} = x_{t+2}$

In this case $I_{t+1} = \frac{x_{t+1}}{x_t} = 1$ and $I_{t+2} = \frac{x_{t+2}}{x_{t+1}} = 1$, and therefore

$$1 + \delta_{t+2,t} = 1 \times 1 = 1$$

This is the correct way to do it .

Index Numbers

⚠ And if we multiplied the indices with the 100?

If instead you used the 100 to multiply:

$$1 + \delta_{t+2,t} = 100 \times 100 = 10000$$

and we get that x grew 100 times! (✗ because we know that $x_t = x_{t+1} = x_{t+2}$)

Index Numbers

We can always transform a chain index number into a fixed base index number.

1. Chose a reference, for example for period 3.
2. Remember: what you want to achieve: $\forall t, I'_t = \frac{x_t}{x_3}$
3. Remember: that we have $I_t = \frac{x_t}{x_{t-1}}$
4. Use what we did just before:

$$I'_t = I_t I_{t-1} \dots I_4 = \frac{x_t}{x_{t-1}} \frac{x_{t-1}}{x_{t-2}} \dots \frac{x_4}{x_3} = \frac{x_t}{x_3} = I'_t$$

Example

Suppose we want to convert this Chain Index into Fix Base index, with the base year in 2021. This will be our 100.

	2021	2022	2023	2024	2025	2026
Price	30	34	37	35	36	39
		$\frac{34}{30} 100$	$\frac{37}{34} 100$	$\frac{35}{37} 100$	$\frac{36}{35} 100$	$\frac{39}{36} 100$
I_{chain}		113.3	108.8	94.6	97.1	108.3
		$100 \frac{113.3}{100}$	$113.3 \frac{108.8}{100}$	$123.3 \frac{94.6}{100}$	$116.7 \frac{97.1}{100}$	$120 \frac{108.3}{100}$
$I_{t 2021}$	100	113.3	123.3	116.7	120	130

Example

Note that, for example in 2025, we have $I_{2025|2021} = 116.7$, which was computed as $123.3 \times \frac{94.6}{100}$. But 123.3 was the base 2021 index of the previous period, which in turn was computed as $113.3 \frac{108.8}{100}$, so in reality:

$$116.7 = 123.3 \frac{94.6}{100} = 113.3 \frac{108.8}{100} \frac{94.6}{100} = 100 \frac{113.3}{100} \frac{108.8}{100} \frac{94.6}{100}$$

If you pay attention, you will see that we have there the product of the *chain Index* in 2022, 2023, and 2024. We divided by 100 because, to multiply, we need to divide by 100 before.

Base Change

In case we have the data, we just need to compute the Index with the base that we want, however, if we do not have the data, and we just have a Fix Base index, we can do it all the same!

Say we have an index with base 0. Maybe we want to change it to base 2.

Base Change

Table 1: Changing Base

Base 0

period	Index	Formula
0	$I_{0 0}$	$\frac{x_0}{x_0}$
1	$I_{1 0}$	$\frac{x_1}{x_0}$
2	$I_{2 0}$	$\frac{x_2}{x_0}$
3	$I_{3 0}$	$\frac{x_3}{x_0}$
...	...	...
t	$I_{t 0}$	$\frac{x_t}{x_0}$

Base 2

period	Index	Formula
0	$I_{0 2}$	$\frac{x_0}{x_2}$
1	$I_{1 2}$	$\frac{x_1}{x_2}$
2	$I_{2 2}$	$\frac{x_2}{x_2}$
3	$I_{3 2}$	$\frac{x_3}{x_2}$
...	...	...
t	$I_{t 2}$	$\frac{x_t}{x_2}$

Base Change

But how can we transform them, if we do not know $x_i \forall i$?

$$I_{t|2} = \frac{x_t}{x_2} = 1 \frac{x_t}{x_2} = \frac{1/x_0}{1/x_0} \frac{x_t}{x_2} = \frac{\frac{x_t}{x_0}}{\frac{x_2}{x_0}} = \frac{I_{t|0}}{I_{2|0}}$$

And that is how we can re-base an index without knowing the value for the underlying variable (x_t)!



Mixing up stuff

Previously we covered how to go from a chain index to a fixed base one. Now we saw how to change the base of a fixed base index. With both things together you can turn a chain index into fixed base one, with base you want!

Example

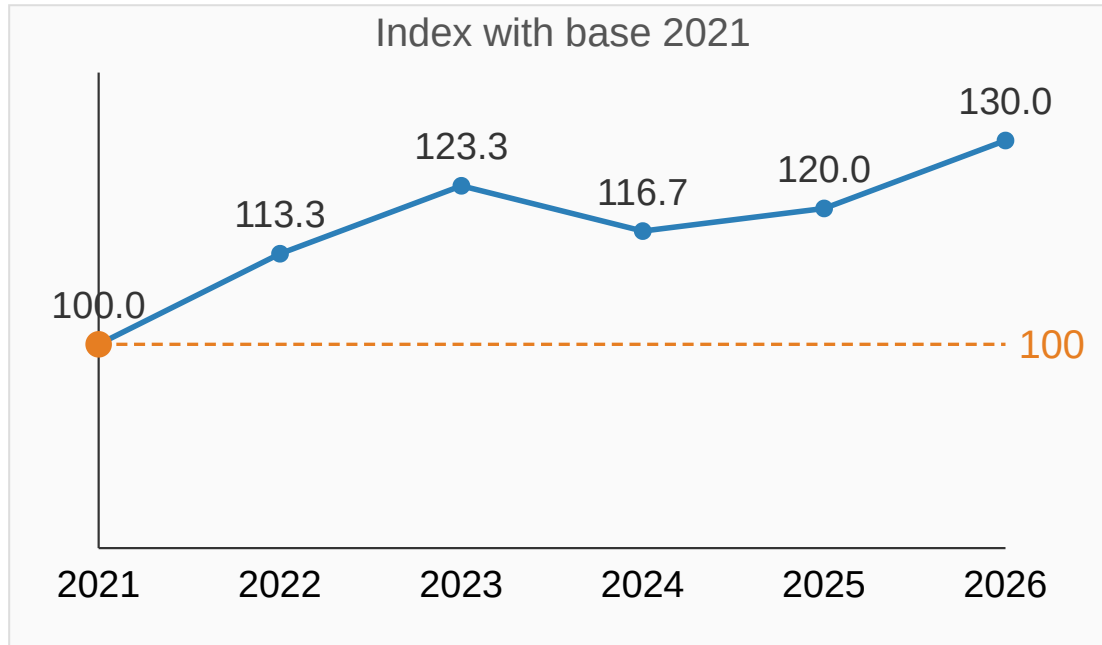
Stock prices

	2021	2022	2023	2024	2025	2026
Price	30	34	37	35	36	39
$I_{t 2021}$	100	113.3	123.3	116.7	120	130
	$\frac{100}{123.3}$	$\frac{113.3}{123.3}$	$\frac{123.3}{123.3}$	$\frac{116.7}{123.3}$	$\frac{120.0}{123.3}$	$\frac{130.0}{123.3}$
$I_{t 2023}$	81.1	91.9	100	94.6	97.3	105.4

New information? Yes! For example, without any computation we know that the price in 2026 was 30% larger than in 2021, and 5.4% larger than in 2023

Does changing the base change the story?

Same prices as above. Move the slider to pick which year is the base, and watch what moves and what does not.



Base year: **2021**



Growth rates

2022: 13.3%

2023: 8.8%

2024: -5.4%

2025: 2.9%

2026: 8.3%

These never change.

What the chart showed

Changing the base **slides the whole curve up or down**: the orange 100 line stays where it is, and a different year gets pinned to it.

But the **shape is untouched**, and every growth rate in the panel stays exactly the same. Re-basing changes the *yardstick*, never the underlying evolution.

If two countries publish the same price series on different bases, they are not disagreeing about inflation. Divide one index by another of the same series and the base cancels, which is precisely the formula we just derived.

Tests for good Simple Index Numbers

We can test the properties for Simple Index Numbers. These represent desirable properties for good index numbers.

1. **Identity test:** If the variable in the current period is the same as in the base period, the index should be 1 (or 100). Of course this implies

$$I_{0|0} = 1$$

.

2. **Proportionality test:** If the variable increases by k , the index should increase by k .

$$x_t = kx_0 \Rightarrow i_{t|0} = k$$

Tests for good Simple Index Numbers

3. **Time Reversal test:** If you exchange the base between two periods, then the index in one base, should be the inverse of the other:

$$i_{t|0} = \frac{1}{i_{0|t}} \Leftrightarrow i_{t|0} i_{0|t} = 1$$

4. **Factor Reversal test:** If we have two index numbers, one for variable x and another for y , then if $z = x \times y$ we should have

$$i_{t|0}^z = i_{t|0}^x \times i_{t|0}^y$$

Tests for good Simple Index Numbers

5. **Circularity test:** The index at t base 0 should be the product of all the link index numbers up to t ,

$$i_{t|0} = i_{t|t-1} \times i_{t-1|t-2} \cdots i_{2|1} \times i_{1|0}$$

This formalizes the relationship between fixed base and chain index numbers.

6. **Homogeneity test:** If the variable, for every t is multiplied by a constant, the index should not be affected.

Tests for good Simple Index Numbers

7. **Well-definedness:** The index number must be meaningful, and as prices and quantities usually are larger than zero, index numbers must be positive. Even further, a good index number should not take the value of ∞ .

Ask yourself, what information are you getting here?

A value of 0 might also be problematic, although it is not totally meaningless. The problem arises when you want to deal with chain index numbers, where a 0 would indeterminate the rest of the chain. Also, you could never use as a base period, a period where the variable took the value of 0.

? Simple Index Numbers · Question 1

A simple price index with base 2020 reads 125 in 2024. This tells you that the price:

A. rose 125%

B. rose 25%

C. is 125 euros

D. nothing, without the base price

✓ **B.** The index is $\frac{x_t}{x_0} \times 100$. A reading of 125 means $\frac{x_t}{x_0} = 1.25$, so the price is 25% **above** the base.

? Simple Index Numbers · Question 2

To turn a chain (link) index into a fixed base index you:

- A. multiply them, in decimal notation
- B. add the link indices
- C. average them
- D. subtract 100 from each

 **A.** Link indices multiply, and you must divide by 100 first, otherwise each extra factor of 100 inflates the answer.



Simple Index Numbers · Question 3

The price of a good was 30 in year 1, 34 in year 2 and 37 in year 3.

Compute the fixed base index with base year 1 for all three years, and the link index for year 3.



Simple Index Numbers · Solution

Fixed base year 1: $I_{1|1} = 100$, $I_{2|1} = \frac{34}{30} \times 100 = 113.3$, $I_{3|1} = \frac{37}{30} \times 100 = 123.3$.

Link index for year 3: $i_{3|2} = \frac{37}{34} \times 100 = 108.8$.

Check: $100 \times \frac{113.3}{100} \times \frac{108.8}{100} \approx 123.3$  (chain the link indices in decimal, never on the 100 scale)

Index Numbers: Composite Index Numbers

Composite Index Numbers

Composite Index Numbers (a.k.a. Aggregate Index Numbers)

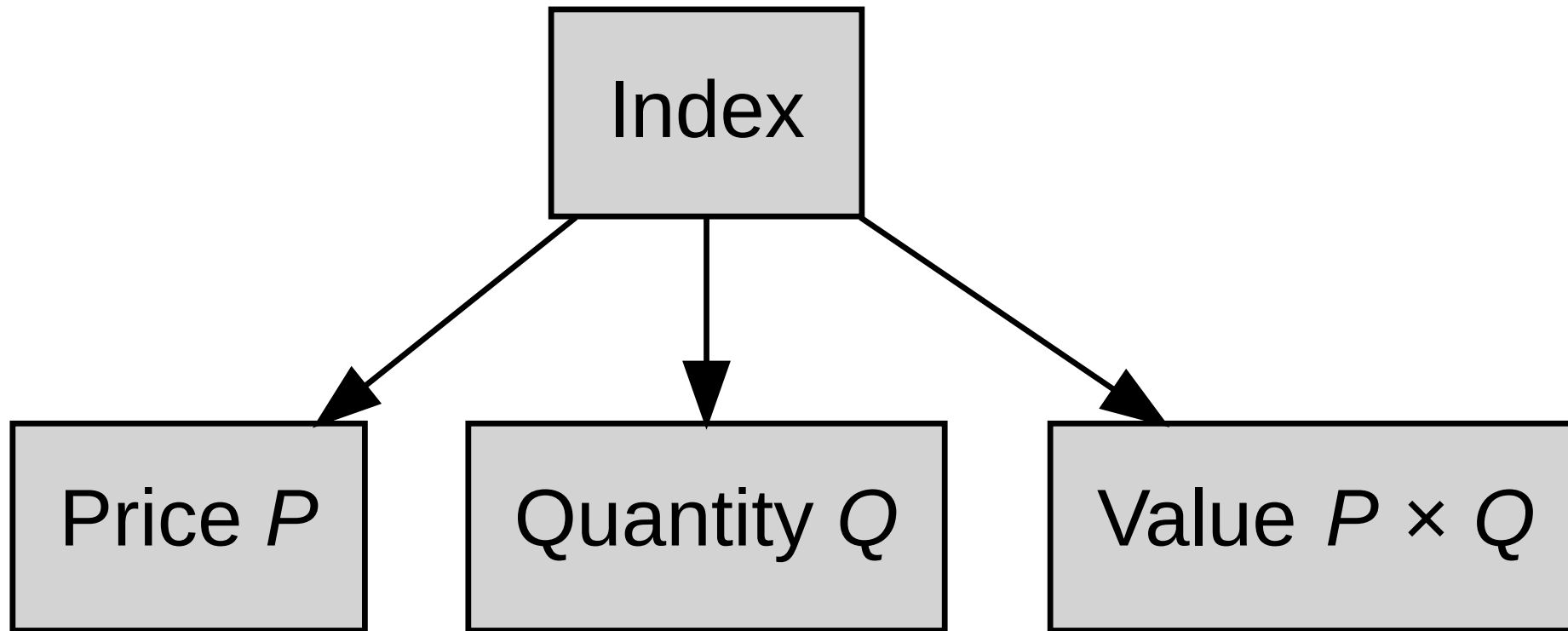
Composite Index Numbers are index numbers that have more than a single underlying variable, or which underlying variable is computed itself as a combination of other variables.

A composite index tracks the combined evolution of multiple phenomena at the same time. It summarizes how a diverse group of elements changes over time.

Example: The performance of a class, measured by the average grade. The average is computed with each individual grade, and therefore if we make an index based on the average of the class, we would have a composite index number.

Composite Index Numbers

This is a great opportunity to introduce another classification for our index numbers:



Composite Index Numbers

Typically, values are collected in nominal terms, *i.e.* **current prices**, which is a *value*.

$$v_t = p_t \times q_t$$

Note that, if δ_{t+1}^v is the growth rate of v_t :

$$v_{t+1} = v_t (1 + \delta_{t+1}^v)$$

and obviously, it is also true that:

$$v_{t+1} = p_{t+1} \times q_{t+1}$$

Composite Index Numbers

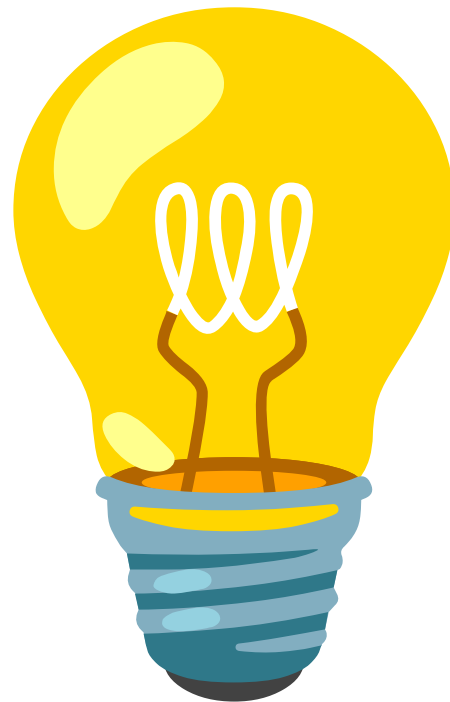
But if we let δ_{t+1}^p be the growth rate of prices, and δ_{t+1}^q the growth rate of quantities we can rewrite v_{t+1} as:

$$v_{t+1} = p_t (1 + \delta_{t+1}^p) q_t (1 + \delta_{t+1}^q)$$

$$v_{t+1} = p_t q_t (1 + \delta_{t+1}^p)(1 + \delta_{t+1}^q)$$

$$v_{t+1} = v_t (1 + \delta_{t+1}^p)(1 + \delta_{t+1}^q)$$

$$v_t (1 + \delta_{t+1}^v) = v_t (1 + \delta_{t+1}^p)(1 + \delta_{t+1}^q)$$



Composite Index Numbers

$$(1 + \delta_{t+1}^v) = (1 + \delta_{t+1}^p)(1 + \delta_{t+1}^q)$$

δ_t^v is what we call **nominal change rate**, while δ_t^q is what we call **real change rate**. We can find out the real change rate (δ^q), if we know the nominal change rate (δ^v), and the prices change rate (δ^p):

$$(1 + \delta_t^q) = \frac{(1 + \delta_t^v)}{(1 + \delta_t^p)}$$

$$\delta_t^q = \frac{(1 + \delta_t^v)}{(1 + \delta_t^p)} - 1$$

Composite Index Numbers

Suppose we have index numbers for v , q , and p , all with base b (remember you can re-base your index numbers easily if they do not share the same base).

$$I_{t|b}^q = \frac{I_{t|b}^v}{I_{t|b}^p}$$

$$\frac{q_t}{q_b} = \frac{\frac{v_t}{v_b}}{\frac{p_t}{p_b}}$$

$$\frac{q_t}{q_b} = \frac{\frac{p_t q_t}{p_b q_b}}{\frac{p_t}{p_b}} = \frac{p_t q_t}{p_b q_b} \frac{p_b}{p_t} = \frac{q_t}{q_b}$$

Example

	2022	2023	2024	2025	2026
Sales Index	99	100	107	110	111
Price Index	95	100	102	106	109

Let's find the quantities index, with base 2023. Immediately we can fix $i_{t|b}^q = 100$.

$$I_{2022}^q = \frac{I_{2022}^v}{I_{2022}^p} = \frac{99}{95} = 1.04$$

Multiplying by 100, $I_{2022}^q = 104$, so quantities decreased from 2022 to 2023!

Example

	2022	2023	2024	2025	2026
Quantities Index	104.21	100	104.9	103.77	101.83

Now we can compute what was the real change rate between 2025 and 2026:

$$\delta_{2026} = \frac{102}{104} - 1 = -0.0187 = -1.87\%$$

Real sales decreased 1.87% from 2025 to 2026.

Example

Let's check:

$$\delta_{2026}^q = \frac{(1 + \delta_{2026}^v)}{(1 + \delta_{2026}^p)} - 1 = \frac{111/110}{109/106} - 1 = 0.9813 - 1$$

$$\delta_{2026}^q = -0.0187 = -1.87\%$$

Baskets

Basket 🛒

Basket are sets of goods, which are used to track the evolution of, for example, a price level.

In economics, we often use price and quantity index numbers to measure how the prices or quantities of a set of goods (a **basket**) change over time.

A very important question: What are the weights that the different goods should have in the basket?

Aggregate Index Numbers

From the family of formulas to create weighted composite index numbers, we will focus on the two most common:

1. **Laspeyres**: uses base-period quantities (prices) as weights.
2. **Paasche**: uses current-period quantities (prices) as weights.

These are used to measure the evolution of *prices* (*quantities*).

Kids' Explanation

Old basket ( *past*):

-  10 apples
-  5 oranges

Old prices ( *past*):

- $p_{\text{apple}} = 1$
- $p_{\text{orange}} = 0.8$

New basket ( *today*):

-  8 apples
-  6 oranges

New prices ( *today*):

- $p_{\text{apple}} = 1.2$
- $p_{\text{orange}} = 1.1$

Let's consider that the **base year is past**.

Kids' Explanation - Prices

Laspeyres - Price:

Use always 🛒_{past}, but relevant 🏷️! Laspeyres Price Index (LPI):

$$\text{🛒}_{past} \mid \text{🏷️}_{past} \quad LPI_{past} = \frac{10 \times 1 + 5 \times 0.8}{10 \times 1 + 5 \times 0.8} = \frac{14}{14} = 1$$

$$\text{🛒}_{past} \mid \text{🏷️}_{today} \quad LPI_{today} = \frac{10 \times 1.2 + 5 \times 1.1}{10 \times 1 + 5 \times 0.8} = \frac{17.5}{14} = 1.25$$

Remember that 1 corresponds to 100, and 1.25 corresponds to 125.

Kids' Explanation - Prices

Paasche - Price:

Use always 🛒_{today}! Paasche Price Index (PPI):

$$\text{🛒}_{today} | \text{🏷️}_{past} \quad PPI_{past} = \frac{8 \times 1 + 6 \times 0.8}{8 \times 1 + 6 \times 0.8} = \frac{12.8}{12.8} = 1$$

$$\text{🛒}_{today} | \text{🏷️}_{today} \quad PPI_{today} = \frac{8 \times 1.2 + 6 \times 1.1}{8 \times 1 + 6 \times 0.8} = \frac{16.2}{12.8} = 1.27$$

Remember that 1 corresponds to 100, and 1.27 corresponds to 127.

In general - Laspeyres

1. For prices, how much the old basket would cost at current prices relative to the **old basket and prices**?

$$LPI = \frac{\sum_{k=1}^m p_t^k q_0^k}{\sum_{k=1}^m p_0^k q_0^k}$$

2. For quantities, how much the current basket would cost at the old prices, relative to the **old basket and prices**?

$$LQI = \frac{\sum_{k=1}^m p_0^k q_t^k}{\sum_{k=1}^m p_0^k q_0^k}$$

Note: denominator is the same.

In general - Paasche

1. For prices, how much more expensive is the **current basket** at current prices, relative to the original prices.

$$PPI = \frac{\sum_{k=1}^m p_t^k q_t^k}{\sum_{k=1}^m p_0^k q_t^k}$$

2. For quantities, how much more the **current basket** costs, relative to what would cost the original basket at current prices.

$$PQI = \frac{\sum_{k=1}^m p_t^k q_t^k}{\sum_{k=1}^m p_t^k q_0^k}$$

Note: numerator is the same.

Conclusions

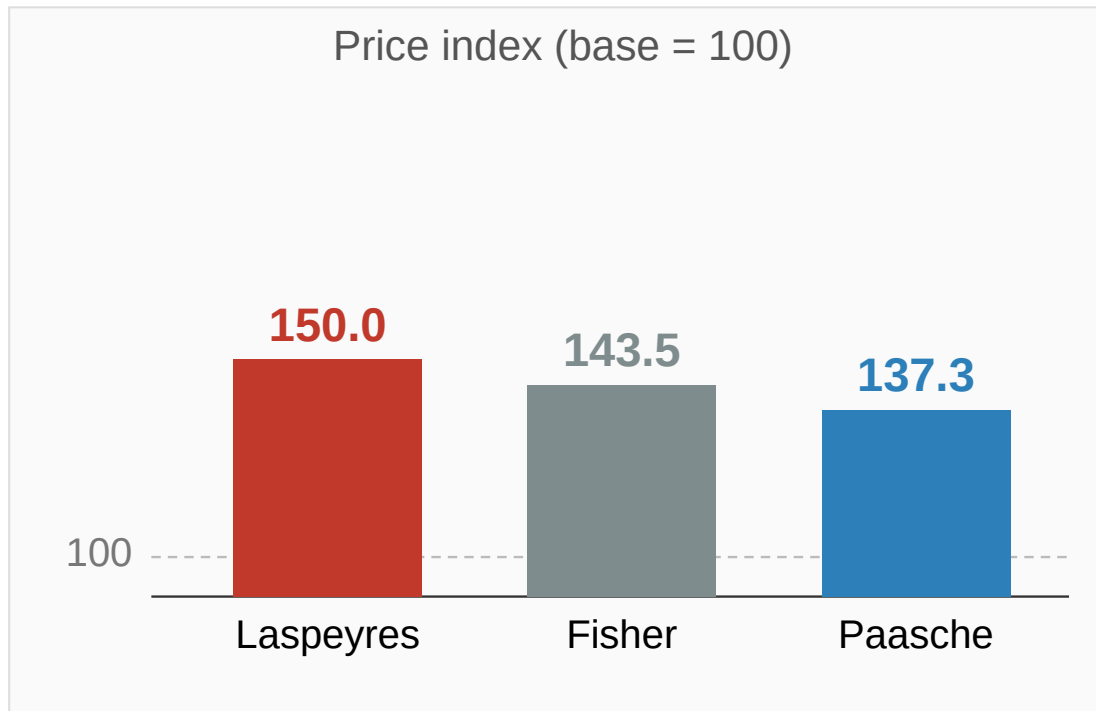
1. In both index numbers, for prices, quantities are constant: All variation comes from prices.
2. In both index numbers, for quantities, prices are constant: All variation comes from quantities.

We tend to use much more often *Laspeyres* because quantities take much longer to be accurately collected than prices. Ergo, national statistics institutes usually use Laspeyres price index to measure the price evolution.

Do the two indices agree?

Two goods, both costing 1 at the base period, both bought 10 times. Now good 1 gets more expensive, and people react by buying less of it.

Left slider: how much more expensive good 1 becomes. **Right slider:** how strongly people substitute away from it.



Price of good 1: **2.00×**



Substitution: **0.50**



Basket

good 1: 10 → 7.1

good 2: 10 → 11.9

Gap $L - P = 12.7$

What the chart showed

With **no substitution** the two indices are identical: the basket never changed, so it does not matter whose basket you weight with.

As substitution grows, **Laspeyres stays put** (it always weights with the *old* basket, which still contains a lot of the good that got dearer) while **Paasche falls** (it weights with the *new* basket, which has already moved away from that good).

So Laspeyres tends to **overstate** the rise in the cost of living, and Paasche tends to **understate** it. The truth is somewhere in between, which is exactly the gap the next index tries to close.

Fisher Index

Some authors argue, however, that the true measure for *cost of life* is in between both metrics. This is captured by the **Fisher Index**:

$$FPI_{t|0} = \sqrt{LPI_{t|0} \times PPI_{t|0}}$$

$$FQI_{t|0} = \sqrt{LQI_{t|0} \times PQI_{t|0}}$$

Example: the data

A basket of 15 goods, observed in 2024, 2025 and 2026. Each cell is what the basket costs with the **row** year's prices and the **column** year's quantities, $\sum_{k=1}^{15} p_{row}^k q_{col}^k$.

	q_{2024}	q_{2025}	q_{2026}
p_{2024}	8562	9210	9920
p_{2025}	9150	9584	
p_{2026}	9657		10013

$$PPI_{2026|2024} = 100 \times \frac{10013}{9920}$$

Example: LPI, PPI and FPI

What are the three indices if we choose 2024 as the base year?

t	2024	2025	2026
$LPI_{t 2024}$	100	106.9	112.8
$PPI_{t 2024}$	100	104.1	100.9
$FPI_{t 2024}$	100	105.5	106.7

$$FPI_{2025|2024} = \sqrt{106.9 \times 104.1}$$

? Composite Index Numbers · Question 1

The Laspeyres price index weights prices using quantities from:

- A. the current period
- B. the base period
- C. the average of both
- D. neither, it uses no quantities

✓ **B.** Laspeyres fixes the **base period** basket, $LPI_{t|0} = \frac{\sum_k p_t^k q_0^k}{\sum_k p_0^k q_0^k}$.

Paasche is the one that uses current quantities.

? Composite Index Numbers · Question 2

The Fisher index is:

- A. the arithmetic mean of Laspeyres and Paasche
- B. the geometric mean of Laspeyres and Paasche
- C. always between 100 and 200
- D. always equal to Laspeyres

☒ **B.** $FPI_{t|0} = \sqrt{LPI_{t|0} \times PPI_{t|0}}.$



Composite Index Numbers · Question 3

For a basket you computed $LPI_{t|0} = 110$ and $PPI_{t|0} = 104$.

Compute Fisher's price index, and explain why it must always fall between the other two.



Composite Index Numbers · Solution

$$FPI_{t|0} = \sqrt{110 \times 104} = \sqrt{11440} \approx 106.96$$

Fisher is the **geometric mean** of the other two, and the geometric mean of two positive numbers always lies between them. So whenever $LPI \neq PPI$, Fisher sits strictly in between, here between 104 and 110.

Inflation

Consumer Price Index

To measure inflation, we use a Laspeyres Price Index, usually known as CPI (Consumer Price Index).

To do that, the National Institute of Statistics (or the equivalent national statistical office) defines a basket, which contains a certain mix of goods and services, that is to be representative of the consumption behavior of the population.

In Portugal , the index has base in 2016, and, as stated above, follows a LPI formulation with small technical adjustments. This index has a monthly frequency, *i.e.* it is computed every month.

Consumer Price Index

To create a price index, you need to follow at least 3 steps.

1. Determine representative basket: Survey families   
2. Survey retail stores to find out sale prices. 
3. Compute the index with the collected data. 
 1. Compute the cost of the  with the prices at the base period.
 2. Compute the cost of the same  with the prices of every successive period.

Application

A Labor union leader is complaining that the average salary in 2022 was 1500, and in 2025 it is 1650. This increase, they say, is not enough to maintain the workers' purchasing power, because of inflation. The data for inflation is the following:

Year	2022	2023	2024	2025
CPI	113	119	122.2	124.8

1. The year with the largest increase in prices was 2024?
2. The average growth rate for the wages between 2022 and 2025 was _____?
3. Is the union leader right?

Application

For the first question, we can compute the chain or link index:

Year	2022	2023	2024	2025
CPI	113	119	122.2	124.8
CPI_{link}	—	105.3	102.7	102.1

So the answer is: The year with the largest price increase was 2023.

Just in case there were doubts, an example:

$$CPI_{link|2024} = \frac{122.2}{119} \times 100 = 102.7$$

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Application

We know salary in 2022 was 1500, and in 2025 1650. If the growth rate was constant, say δ_w we would have:

$$1650 = 1500(1 + \delta_w)^3$$

Or conversely

$$\frac{1650}{1500} = (1 + \delta_w)^3$$

or

$$(1.1)^{1/3} = 1 + \delta_w$$

$$1.0323 - 1 = \delta_w \Rightarrow \delta_w = 0.0323 = 3.23\%$$

Summarizing

Remember this relationship we saw before?

$$I_{t|b}^q = \frac{I_{t|b}^v}{I_{t|b}^p}$$

Real equals nominal over CPI.

This, dividing by CPI a nominal quantity, is known as *to deflate* by the CPI.

About the Test for Factor Reversal for Composite Index Numbers

Note that neither Laspeyres nor Paasche pass the test for **factor reversal** ([see here](#)),

$$VI_{t|0} \neq LPI_{t|0} \times LQI_{t|0} \quad VI_{t|0} \neq PPI_{t|0} \times PQI_{t|0}$$

But we can verify that:

$$LPI_{t|0} \times PQI_{t|0} = IV_{t|0} \quad PPI_{t|0} \times LQI_{t|0} = IV_{t|0}$$

About the Test for Factor Reversal for Composite Index Numbers

(... continuation)

Example (do the rest as homework):

$$LPI_{t|0} \times PQI_{t|0} = \frac{\sum_{k=1}^m p_t^k q_0^k}{\sum_{k=1}^m p_0^k q_0^k} \frac{\sum_{k=1}^m p_t^k q_t^k}{\sum_{k=1}^m p_t^k q_0^k} = \frac{\sum_{k=1}^m p_t^k q_t^k}{\sum_{k=1}^m p_0^k q_0^k}$$

Note that this last item is exactly $IV_{t|0}$.

You can verify, using the same procedure, that Fisher's index does indeed pass the factor reversal test.

About the Factor Reversal Test for Composite Index Numbers

$$\begin{aligned}
 FPI_{t|0} \times FQI_{t|0} &= \\
 \sqrt{LPI_{t|0} \times PPI_{t|0}} \times \sqrt{LQI_{t|0} \times PQI_{t|0}} \\
 &= \sqrt{LPI_{t|0} \times PPI_{t|0} \times LQI_{t|0} \times PQI_{t|0}} \\
 &= \sqrt{LPI_{t|0} \times PQI_{t|0} \times LQI_{t|0} \times PPI_{t|0}} \\
 &= \sqrt{IV_{t|0} \times IV_{t|0}} = \sqrt{IV_{t|0}^2} = IV_{t|0}
 \end{aligned}$$

About the Time Reversal Test for Composite Index Numbers

Laspeyres and Paasche do not pass the time Reversal test, let's see an example:

$$LPI_{t|0} \times LPI_{0|t} = \frac{\sum_{k=1}^m p_t^k q_0^k}{\sum_{k=1}^m p_0^k q_0^k} \frac{\sum_{k=1}^m p_0^k q_t^k}{\sum_{k=1}^m p_t^k q_t^k} \neq 1$$

You can do the same for Paasche, and for the quantities index numbers.

About the Time Reversal Test for Composite Index Numbers

Fisher on the other hand...

$$\begin{aligned}
 FPI_{t|0} \times FPI_{0|t} &= \sqrt{LPI_{t|0} PPI_{t|0}} \sqrt{LPI_{0|t} PPI_{0|t}} \\
 &= \sqrt{\frac{\sum_{k=1}^m p_t^k q_0^k}{\sum_{k=1}^m p_0^k q_0^k} \frac{\sum_{k=1}^m p_t^k q_t^k}{\sum_{k=1}^m p_0^k q_t^k}} \sqrt{\frac{\sum_{k=1}^m p_0^k q_t^k}{\sum_{k=1}^m p_t^k q_t^k} \frac{\sum_{k=1}^m p_0^k q_0^k}{\sum_{k=1}^m p_t^k q_0^k}} \\
 &= \sqrt{\frac{\sum_{k=1}^m p_t^k q_0^k}{\sum_{k=1}^m p_0^k q_0^k} \frac{\sum_{k=1}^m p_t^k q_t^k}{\sum_{k=1}^m p_0^k q_t^k} \frac{\sum_{k=1}^m p_0^k q_t^k}{\sum_{k=1}^m p_t^k q_t^k} \frac{\sum_{k=1}^m p_0^k q_0^k}{\sum_{k=1}^m p_t^k q_0^k}}
 \end{aligned}$$

About the Time Reversal Test for Composite Index Numbers

Fisher on the other hand...

$$\begin{aligned}
 FPI_{t|0} \times FPI_{0|t} &= \sqrt{LPI_{t|0} PPI_{t|0}} \sqrt{LPI_{0|t} PPI_{0|t}} \\
 &= \sqrt{\frac{\sum_{k=1}^m p_t^k q_0^k}{\sum_{k=1}^m p_0^k q_0^k} \frac{\sum_{k=1}^m p_t^k q_t^k}{\sum_{k=1}^m p_0^k q_t^k}} \sqrt{\frac{\sum_{k=1}^m p_0^k q_t^k}{\sum_{k=1}^m p_t^k q_t^k} \frac{\sum_{k=1}^m p_0^k q_0^k}{\sum_{k=1}^m p_t^k q_0^k}} \\
 &= \sqrt{\frac{\sum_{k=1}^m p_t^k q_0^k}{\sum_{k=1}^m p_0^k q_0^k} \frac{\sum_{k=1}^m p_t^k q_t^k}{\sum_{k=1}^m p_0^k q_t^k} \frac{\sum_{k=1}^m p_0^k q_t^k}{\sum_{k=1}^m p_t^k q_t^k} \frac{\sum_{k=1}^m p_0^k q_0^k}{\sum_{k=1}^m p_t^k q_0^k}} \\
 FPI_{t|0} \times FPI_{0|t} &= 1
 \end{aligned}$$

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About the Circularity Test for Composite Index Numbers

None, not even Fisher's Index, passes the circularity test.

Homework: Verify it.

Example

Public Transportation	Year 1		Year 2	
	Trips	Price per trip	Trips	Price Index (Base: year 1)
Bus	1,763,521	1.15	1,875,345	108.70
Metro	3,148,350	1.2	3,396,138	112.50

Example

1. According to LPI for year 2, prices for public transportation grew 11.17%.
2. Real change, according to Paasche, for public transportation was 7.35%.
3. The composite index for change in expenditures with public transportation in year 2 compared the expenditure in year 1 is what type of change? (real, nominal, quantity, prices?) How much was it?

Example

According to LPI for year 2, prices for public transportation grew 11.17%.

First, we need to find p_2 for bus and metro.

$$108.7 = \frac{p_2^{bus}}{1.15} \times 100$$

$$\text{Or } p_2^{bus} = 1.25$$

$$112.5 = \frac{p_2^{metro}}{1.2}$$

$$\text{Or } p_2^{metro} = 1.35$$

Example

	Year 1	Year 2
Public Transportation	N Trips	Ticket price
Bus	1,763,521	1.15
Metro	3,148,350	1.2

Example

$$LPI_{2|1} = \frac{\sum_{k=1}^m p_2^k q_1^k}{\sum_{k=1}^m p_1^k q_1^k}$$

$$LPI_{2|1} = \frac{1,763,521 \times 1.25 + 3,148,350 \times 1.35}{1,763,521 \times 1.15 + 3,148,350 \times 1.2}$$

$$LPI_{2|1} = 1.1117$$

The *basket* in year 2 is 11.17% more expensive than in year 1.

$$LPI_{2|1} \times 100 - 100 = 11.17\%$$

Example

Real change, according to Paasche, for public transportation was 7.35%.

$$PQI_{2|1} = \frac{\sum_{k=1}^m p_2^k q_2^k}{\sum_{k=1}^m p_2^k q_1^k}$$

$$PQI_{2|1} = \frac{1,875,345 \times 1.25 + 3,396,138 \times 1.35}{1,763,521 \times 1.25 + 3,148,350 \times 1.35}$$

$$PQI_{2|1} = 1.0735$$

Indeed, according to the Paasche index, in year 2 the amount of public transportation trips was 7.35% larger than in year 1.

$$PQI_{2|1} \times 100 - 100 = 7.35\%$$

Example

The composite index for change in expenditures with public transportation in year 2 compared the expenditure in year 1 is what type of change? (real, nominal, quantity, prices?) How much was it?

The value index for public transportation is a *nominal* index (price times quantity).

Remember:

$$VI_{t|0} = LPI_{t|0} \times PQI_{t|0}$$

Example

$$VI_{2|1} = LPI_{2|1} \times PQI_{2|1} = 1.1117 \times 1.0735 \approx 1.1934$$

The value of public transportation trips grew 19.34% from year 1 to year 2. This includes the effect of an increase in prices and quantities.

? Inflation · Question 1

The Consumer Price Index is constructed as a:

A. Paasche quantity index

B. simple index number

C. Fisher index

D. Laspeyres price index

 **D.** The basket is fixed at a base period, which is exactly a Laspeyres price index (with small technical adjustments).

? Inflation · Question 2

Your nominal wage rises 10% while the CPI rises 12%. Your purchasing power:

A. rose about 10%

B. did not change

C. rose 2%

D. fell about 1.8%

✓ **D.** Real change = $\frac{1.10}{1.12} - 1 \approx -0.0179$, a fall of about 1.8%. Prices grew faster than the wage.



Inflation · Question 3

A worker earned 1500 three years ago and earns 1650 today. Over the same period the CPI went from 100 to 112.

Did the worker gain or lose purchasing power, and by how much?



Inflation · Solution

Nominal wage growth: $\frac{1650}{1500} - 1 = 0.10$, that is 10%. Price growth: $\frac{112}{100} - 1 = 0.12$, that is 12%.

$$\delta^q = \frac{1 + \delta^v}{1 + \delta^p} - 1 = \frac{1.10}{1.12} - 1 \approx -0.0179$$

The worker **lost** about 1.8% of purchasing power: the wage grew, but prices grew faster. 